# 10/29 Recitation

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

   (a) Compute the determinant starting with a cofactor expansion along the first row

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 1(-1)^2 \begin{vmatrix} 0 & 4 \\ 1 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} -1 & 4 \\ 1 & 1 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -4 + 10 - 3 = 3$$

   (b) Compute the determinant starting with a cofactor expansion along the second column

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 2(-1)^3 \begin{vmatrix} -1 & 4 \\ 1 & 1 \end{vmatrix} + 1(-1)^5 \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} = 10 - 7 = 3$$

   (c) Compute the determinant using row reduction

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 7 \\ 0 & -1 & -2 \end{vmatrix} \text{ (Replace R2 with R2+R1 and R3 with R3-R1)}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 7 \\ 0 & 0 & 3/2 \end{vmatrix} \text{ (Replace R3 with R3+1/2R2}$$

$$= 1(2)(3/2) = 3$$

2. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. In 10/24 recitation, you used Julia to find the formula

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}.$$

   Derive this formula using the cofactor expansion along the first row.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^4 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

3. Recall that if $A$ is invertible, then $A^{-1} = \frac{1}{\det A} C^T$, where $C$ is the cofactor matrix. Let's prove this.

   (a) Show that the $(i,j)-$entry of $AC^T$ is $a_{i1}C_{j1} + \cdots + a_{in}C_{jn}$.

   the$i$th row of $A$ is $\begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix}$ and the $j$th column of $C^T$ is $\begin{bmatrix} C_{j1} \\ \vdots \\ C_{jn} \end{bmatrix}$. So the $(i,j)$ entry of $AC^T$ is

   $$\begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} C_{j1} \\ \vdots \\ C_{jn} \end{bmatrix} = a_{i1}C_{j1} + \cdots + a_{in}C_{jn}$$

   (b) Show that if $i = j$, then $a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = \det A$. Show that if $i \neq j$, then $a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = 0$ by interpreting this expression as the determinant of a different matrix.

   Let $i = j$. Then $a_{i1}C_{i1} + \cdots + a_{in}C_{in}$ is the cofactor expansion of $A$ along the $i$th row; hence it is simply $\det A$. Now let $i \neq j$. Then $a_{i1}C_{j1} + \cdots + a_{in}C_{jn}$ can be viewed as the cofactor expansion along the $j$th row of the matrix $A'$ obtained from $A$ by replacing the $j$th row with $\begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix}$. Since $A'$ has two identical rows, $\det A' = 0$; hence $a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = \det A' = 0$.

   (c) Show that $AC^T = (\det A)I$ and deduce that $A^{-1} = \frac{1}{\det A} C^T$.

   By part (b), we have that the $(i,j)$-entry of $AC^T$ is 0 whenever $i \neq j$. When $i = j$, the entry is $\det A$. Hence $AC^T = (\det A)I$. Now multiplying both sides by $A^{-1}$, we obtain $C^T = A^{-1}((\det A)I) = (\det A)(A^{-1}I) = (\det A)A^{-1}$. Hence $(\det A)A^{-1} = C^T$ or $A^{-1} = \frac{1}{\det A} C^T$.

4. True/False. If a statement is true, prove it. If it is false, provide counterexample.

   (a) $\det(A + B) = \det A + \det B$

   False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
   Then $\det(A + B) = \det I = 1$, whereas $\det A + \det B = 0 + 0 = 0$.

(b) The determinant of any projection matrix (other than the identity) is 0.

True. Let $P$ be an $n \times n$ projection matrix; then the associated linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ projects vectors in $\mathbb{R}^n$ onto $C(P)$. Since $P \neq I$, $\dim C(P) < n$. It follows that this map is not onto; hence $P$ is not invertible and so $\det P = 0$.

5. **Looking Ahead:** Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $\lambda \in \mathbb{R}$. Find $\det(A - \lambda I)$ and solve the equation $\det(A - \lambda I) = 0$ for $\lambda$.

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} \\
&= (2-\lambda)(-1)^2 \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 1 & 1 \\ 0 & 2-\lambda \end{vmatrix} \text{ (R1 cofactor expansion)} \\
&= (2-\lambda)[(2-\lambda)^2 - 1] - (2-\lambda) \\
&= (2-\lambda)[(2-\lambda)^2 - 2] \\
&= (2-\lambda)(\lambda^2 - 4\lambda + 2)
\end{aligned}$$

We now need to solve $(2-\lambda)(\lambda^2 - 4\lambda + 2) = 0$. Since one of the factors must be zero, we must solve $\lambda - 2 = 0$ and $\lambda^2 - 4\lambda + 2 = 0$. The first gives $\lambda = 2$ and the second gives $\lambda = 2 \pm \sqrt{2}$ (with help of the quadratic formula).